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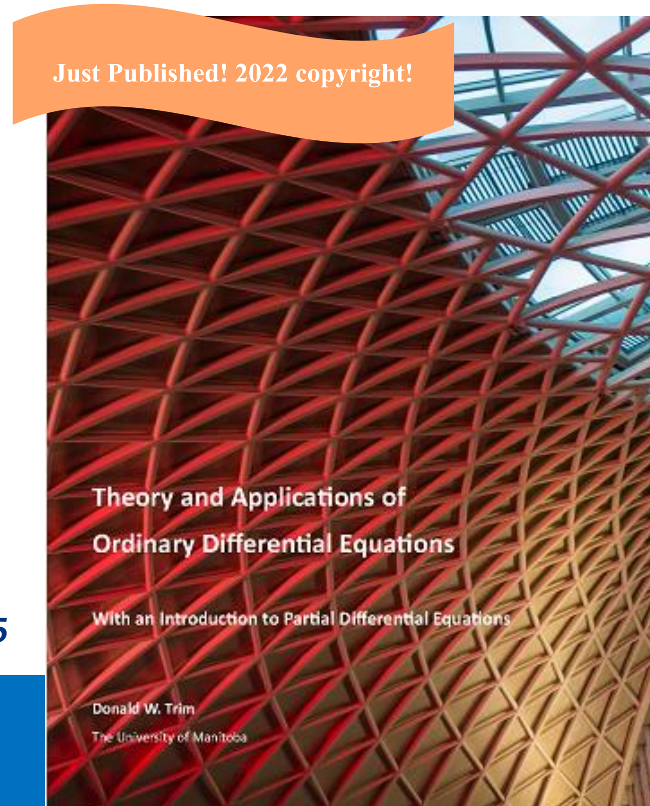

Theory and Applications of Ordinary Differential Equations

with an Introduction to Partial Differential Equations

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This edition is a well-balanced blend of theory, techniques, and applications.

The author has written the text so that instructors can choose to emphasize whichever of these three aspects of differential equations is appropriate for the course.

- The author's writing style will appeal to students. It replaces the stylized language akin to technical papers and more advanced texts with smoothly flowing discussions that students will find easy to follow. The author seems to know when and what questions students will ask, and provides the answer at the appropriate time.

- The text has plenty of exercises ranging from the routine (to test understanding of basic subject matter) to challenging (for the enterprising student). Exercises avoid the long and involved applications that few, if any, students are likely to investigate.

- Partial differential equations is a vast area of study. The author gives thorough treatments to the most prominent topics in any introductory course on partial differential equations. Excerpts are replaced by complete discussions, leaving the reader with a sense of mastery of each topic. Included are separation of variables on homogeneous and nonhomogeneous problems in Cartesian coordinates, regular and singular Sturm-Liouville systems, and separation in polar, cylindrical, and spherical coordinates.

- The complete solutions manual has been written and prepared by the author so students can be assured that all exercises can be solved by techniques discussed in the text.

Preface, Table of Contents and Sample Chapter

TABLE OF CONTENTS

To the Instructor

Areas of Study

The first ten chapters of this text contain detailed discussions of ordinary differential equations (ODEs); the last four chapters provide an introduction to partial differential equations (PDEs). There are three areas of study in differential equations: theory (existence, uniqueness, and properties of solutions), solving techniques, and applications. Every author makes a choice as to which of these three aspects of differential equations is paramount. We believe that all three areas are important. When students finish a first course in differential equations, we recommend that they know when solutions to differential equations exist, when solutions are unique, and properties that they possess; be familiar with techniques for solving differential equations (or approximate solutions in the event that they seem intractable); and be able to apply differential equations in a variety of situations. Section 1.3 is devoted to existence, uniqueness, and properties of solutions to initial-value problems associated with first-order differential equations. Parts of Sections 2.1, 4.2, 6.4, and 7.1 also contain discussions on existence, uniqueness, and properties of solutions, and much of Chapter 8 is theoretical. As evidenced by their titles, all of Chapters 3, 5, and 12 are devoted to applications of differential equations as are parts of Sections 6.5, 6.6, 7.9, 8.5, 14.4, and 14.6. In addition, many examples and exercises throughout the book contain applications of differential equations from a multitude of fields. Chapters 2, 4, 7, and 13 are devoted to techniques for finding algebraic solutions to ODEs and PDEs. Chapters 9 and 10 contain material on how to approximate solutions to differential equations using infinite series and numerical methods when algebraic solutions are unavailable.

In order to create flexibility in course construction, we have tried to make chapters, and sections within chapters, as modular as possible. We suggest that Chapters 1–5, or parts thereof, should be early material in most courses. After this, the order in which Chapters 6–10 are covered is at your discretion.

Prerequisites

Students must have completed two semesters of single-variable calculus; they must not only be able to differentiate and integrate standard functions, but also understand the meaning of these processes. A course in multivariable calculus is beneficial, especially for PDEs. A course in linear algebra is also recommended, although not a strict requirement. Some of the concepts developed for linear differential equations (linear independence of solutions and Wronskians) have their birth in linear algebra. In addition, proofs of theoretical results for linear equations often draw upon linear algebra. Finally, two techniques for solving systems of ODEs use eigenvalues and eigenvectors of matrices, and generalized eigenvectors. Phase plane analysis in Chapter 8 is heavily dependent on this material. In the event that your course covers material in Chapter 8, we have included an appendix on eigenvalues, eigenvectors, and generalized eigenvectors for students with deficiencies in this topic.

Calculators and Computers

There are many mathematical programs for calculators and computers that can be of benefit to anyone dealing with differential equations. Mathematica, Maple, and MatLab, to name a few, can integrate and differentiate functions, perform labourious calculations that detract from the flow of ideas, and have subroutines that solve certain types of differential equations. As an instructor, you must decide whether these programs are a major component of the course, or whether to make students aware of their existence and capabilities, but not make them a requirement. We have not made them an essential part of the text; we recommend that students use them to perform mundane calculations (such as finding eigenvalues and eigenvectors of matrices), checking a proposed solution to a differential equaton (by doing the differentiations), or graphing a solution to a differential equation (to see its properties). On the other hand, calculators or computers are somewhat essential for numerical methods to approximate solutions to differential equations in Chapter 10. They can be programmed to perform the multitude of repetitive calculations.

Exercises

There is an abundance of exercises ranging from routine to challenging. Early exercises in each section test familiarity with subject material. Difficulty with these exercises indicates a lack of understanding of fundamental ideas. Students should be advised to reread explanations and examples in the section. If trouble still persists, they should seek outside help such as help centres or you personally. Later exercises test mastery of material; they require more complicated calculations, thinking about concepts of the section in a novel way, bringing in material from previous sections, or a combination of these.

Chapter Summaries and Review Exercises

Concluding each chapter is a summary of ideas, formulas, and procedures, and a set of review exercises. These exercises are important in assessing student understanding of the chapter as a whole. For instance, in Section 2.2 which deals with separable equations, students know that each differential equation is separable, and proceed to solve the problem on this basis. In review exercises, however, students do not know, a priori, the type of the differential equation. An extra step is added to the solution; first determine the type of the differential equation, and then proceed with the appropriate technique for that type.

Solutions Manual

A solutions manual is available containing detailed solutions of all exercises. It was written by the author so that students can be assured that techniques are those covered in the text.

In what follows we give brief descriptions of the contents of each chapter, and where appropriate, explain the rationale for our approach.

Chapter 1 — Introduction to Differential Equations

Students are introduced to the language of differential equations, what a differential equation is, and what it means to be a solution of a differential equation on an open interval. We choose to discuss differential equations on open intervals in order to avoid one-sided derivatives at end points of closed intervals. Students become familiar with types of solutions — implicitly defined, particular, n-parameter families, general, and singular. Direction fields are introduced at an early stage urging students to think about first-order differential equations geometrically as well as algebraically. Direction fields can be used to glean information about solutions before any attempt is made to find solutions, and they can be used to check whether proposed solutions are reasonable. The chapter also introduces students to the theory of differential equations by discussing existence, uniqueness, and properties for solutions to first-order initial-value problems.

Chapter 2 — First-order Differential Equations

This is the first chapter devoted to techniques for solving differential equations. Methods include separation, integrating factors, homogeneous, exact, Bernoulli, and Riccati equations. Since sections are independent of each other, you can choose techniques useful to students in your course. For instance, it may be appropriate to cover only linear and separable equations since the majority of applications give rise to these types of equations. Should a particular application that you wish to cover require one of the other techniques, you can teach it as part of this chapter, or, return to it later in conjunction with the application. We stress differences between linear and nonlinear equations. There is also a section on second-order equations reducible to a pair of first-order equations.

Chapter 3 — Applications of First-order Equations

Chapter 3 introduces readers to the third aspect of differential equations, applications that give rise to first-order differential equations. This is where the process of mathematical modelling begins in earnest. Students get an introduction to the ideas of mathematical modelling in some of the examples and exercises of Chapter 2, but this chapter immerses them in modelling. By scaning the table of contents, students can appreciate the plethora and variety of applications, and this is only for first-order equations.

Chapter 4 — Linear Differential Equations

This chapter provides a thorough treatment of linear equations. It contains the theory of linear equations and techniques to solve them. We often use second-order equations to introduce ideas, but general results are stated and proved for n^{th}-order equations. We feel that students at this level do not need separate chapters for second- and n^{th}-order equations. Four techniques are discussed for finding particular solutions of nonhomogeneous problems, undetermined coefficients (perhaps the most widely used), annihilators (an alternative for arriving at the form of a particular solution for undetermined coefficients), variation of parameters (not being subject to the same restrictions as undetermined coefficients), and operators (popular with engineering students).

Chapter 5 — Applications of Linear Differential Equations

Vibrating mass-spring systems which form the foundation for more complicated vibration problems are given special treatment. The bulk of the chapter is devoted to the topic for a single mass. Section 7.9 extends discussions to multiple mass-spring systems. Because discussions of LCR circuits would parallel those for vibrating mass-spring systems, they are treated in less detail. Electrical networks are treated in Section 7.9. We discuss undamped resonance leading to unbounded oscillations, and damped resonance where oscillations cannot be unbounded, but can be detrimental or beneficial. Heaviside unit step functions are introduced to represent piecewise continuous loadings on beams, but we point out that applications involving such functions are best handled by Laplace transforms.

Chapter 6 — Laplace Transforms

Laplace transforms, which are essential to many branches of engineering, are treated extensively. Although Laplace transforms are particularly useful for solving initial-value problems, we demonstrate that they can also be applied to boundary-value problems, and can generate general solutions to linear differential equations. Representation of instantaneous forces (in time), point forces (in space), point masses, point charges, voltage spikes, bulk additions of solutes in mixing problems, etc., are represented by Dirac-delta functions, and Laplace transforms are the best way to handle these functions in the context of differential equations.

Chapter 7 — Systems of Diferential Equations

Many applications give rise to systems of differential equations in many dependent variables as opposed to a single differential equation in one dependent variable. They can be linear and they can be nonlinear. We use elimination (or operators), Laplace transforms, eigenvalues and eigenvectors of matrices, variation of parameters, and decoupling to solve linear systems. They are applied to mixing problems among multiple tanks, multiple vibrating mass-spring systems, electrical networks, and systems akin to radioactive decay series.

Chapter 8 — Nonlinear Differential Equations and Systems

Discussions are confined to autonomous differential equations and autonomous systems. Direction fields are replaced by tangent fields, and applied to two classical ecological models, predator-prey, and competitive-hunter. Critical points of linear systems are categorized, and then applied to linear systems that approximate nonlinear systems. These analyses are instrumental in displaying solutions geometrically in the phase plane. Eigenvalues and eigenvectors of matrices are essential to our approach to this topic. Results are applied to the above models as well as other physical problems that give rise to linear and nonlinear systems.

Chapter 9 — Series Solutions of Differential Equations

In spite of the fact that Taylor series and Maclaurin series are power series, and conversely, every power series is the Taylor or Maclaurin series of its sum, the method by which we derive the Taylor or Maclaurin series for the solution of a differential equation is very different from how we arrive at a power series solution. Processes are different, but results are identical. Frobenius series solutions for differential equations are discussed in detail with an abundance of examples and exercises. Series solutions are derived for the differential equations of Bessel, Legendre, Chebyshev, Hermite, and Laguerre. Important properties of resulting functions (generating functions, orthogonality, recurrence formulas, and Rodrigues' formulas) are included.

Chapter 10 — Numerical Solutions of Differential Equations

When a differential equation is so intractable as to not yield analytic solutions, and infinite series solutions are also not available, it may be necessary to resort to numerical approximations to solutions. In spite of the fact that numerical procedures are available for second- and higher-order differential equations, we consider only numerical techniques for first-order equations and first-order systems. We do so because higher-order equations and systems can be rewritten as first-order systems. We consider predictor and predictor-corrector methods, Euler, improved

Euler, Runge-Kutta, and Adams-Moulton. It can be perilous to accept numerical approximations without some kind of justification. One possibility is to plot approximations on the direction field to see if there is a reasonable fit. Many instructors prefer an early introduction to numerical approximations. This is possible. Sections 1, 2, and 3 which deal with numerical techniques for a single first-order differential equation can be covered at any time thoughout your course; they do not depend on the first nine chapters. Section 4, on the other hand, does require the introduction of systems of differential equations in Chapter 7.

Chapter 11 — Fourier Series

Fourier series represent periodic functions, or functions that can be extended periodically, in terms of sines and cosines. Their advantage over power series is that the function need not be continuous, but it does need to be piecewise continuous. We discuss full Fourier series, Fourier sine series for odd functions, and Fourier cosine series for even functions. Fourier series are basic for most beginning courses in partal differential equations; they also lead to more advanced techniques in future courses.

Chapter 12 — Partial Differential Equations of Mathematical Physics

In this chapter, we do some advanced mathematical modelling by developing the one-, two-, and three-dimensional heat conduction equations based on Faraday's law of heat conduction, the one-dimensional wave equation for transverse vibrations of strings and longitudinal vibrations of bars, the two-dimensional wave equation for transverse vibrations of membranes, and the Laplace and Poisson equations for electrostatic potential.

Chapter 13 — Separation of Variables

Separation of variables is likely the first technique that most students learn for solving second-order PDEs. We use it to find Fourier series representations for simple, homogeneous problems expressed in Cartesian coordinates. We also extend it to three types of nonhomogeneous problems. Firstly, nonhomogeneities associated with Laplace's equation are handled by splitting the nonhomgeneous problem into homogeneous ones. Secondly, time-independent nonhomogeneities in heat conduction problems are handled by splitting off the steady-state solution. Likewise, time-independent nonhomogeneities in vibration problems split off static deflections. Finally, when nonhomogeneities are time-dependent, a method similar to variation of parameters is introduced. Separation of variables is applied to Laplace's equation in polar coordinates. It is also applied to a few problems in cylindrical and spherical coordinates, but the main discussion of separation in these coordinate systems is delayed until Chapter 14 when singular Sturm-Liouville systems have been discussed.

Chapter 14 — Boundary-value Problems and Sturm-Liouville Systems

Section 14.2 is devoted to Sturm-Liouville (SL) systems, the basis for all separation of variables for PDEs in any number of variables and in Cartesian, polar, cylindrical, and spherical coordinates. Regular SL systems arise in heat conduction, vibration, and potential problems cast in Cartesian coordinates. We stress the benefits of orthonormal eigenfunctions, and demonstrate that Fourier sine and cosine series are but special cases of SL theory. We discuss singular SL systems associated with Bessel and Legendre differential equations and apply them to problems in polar, cylindrical, and spherical coordinates.

The author would appreciate being made aware of errors in the text or solutions manual, typographical, reasoning, referencing, etc.

CHAPTER 1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

In physical systems, and even in systems that are not physical, a change in one quantity often precipitates changes in other quantities. Relationships among the rates at which these changes take place lead to what are called *differential equations*, and the number of areas of applied mathematics that give rise to differential equations is truly amazing. The fact that they serve as mathematical models not only in the physical sciences of engineering, physics, and chemistry, but also in less traditional areas such as economics, medicine, and music, indicates why they are worthy of our study. In this chapter we give various classifications of differential equations along with some simple examples to illustrate the direction of investigations in the remainder of the text.

1.1 Classification of Differential Equations

You have likely solved hundreds, even thousands, of equations in your mathematical studies. You have solved algebraic equations like $x^3 + 5x^2 - x - 5 = 0$ for values $x = -5, -1, 1$. You have also solved equations like $x^3y + x^2 = 5$ and $e^y - e^{-y} = 2x$ that define y implicitly as functions of x for explicit definitions of the functions, namely, $y = (5 - x^2)/x^3$ and $y = \ln(x + \sqrt{x^2+1})$. Differential equations must also be solved for functions, but unlike the above equations, differential equations contain one or more derivatives of the unknown functions. This makes them much more formidable, but at the same time, far more interesting. A **differential equation** then is an equation containing derivatives of unknown functions that must be solved for the functions. For example, each of the following equations is a differential equation in y as a function of x, except in equation 1.1e, where x is a function of time t:

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{C}\right), \quad (k \text{ and } C \text{ constants}), \tag{1.1a}$$

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad (k \text{ a constant}), \tag{1.1b}$$

$$xy'' + y' + xy = 0, \tag{1.1c}$$

$$\frac{d^4y}{dx^4} - k^4y = 1, \quad (k \text{ a constant}), \tag{1.1d}$$

$$m\ddot{x} + \beta\dot{x} + kx = \sin t, \quad (m,\ \beta, \text{ and } k \text{ constants}), \tag{1.1e}$$

$$y^{(10)}(x) + 4xy^{(5)}(x) - 2x\frac{dy}{dx} = \sin x. \tag{1.1f}$$

Equation 1.1a is used to determine sizes of populations which follow the *logistic model* (Section 3.3); equation 1.1b describes shapes of hanging cables (Section 2.8); equation 1.1c, called Bessel's differential equation of order zero, is found in heat flow and vibration problems (Section 9.9); and equation 1.1d is used to determine deflections of beams (Section 5.5). Engineers often indicate derivatives with respect to time with dots above a function as in equation 1.1e. One dot in $\dot{x}$ represents dx/dt, and two dots in $\ddot{x}$ means d^2x/dt^2. This equation arises in Section 5.1 when we study vibrating mass-spring systems. Equation 1.1f has no applications that we are aware of. We have listed it to introduce the notation $y^{(10)}(x)$ which is sometimes shortened to $y^{(10)}$; it represents the tenth derivative of the function $y(x)$ with respect to x. A superscript enclosed in parentheses on a function always indicates a derivative of the function, not a power of the function. Likewise then, $y^{(5)}(x)$ is the fifth derivative of $y(x)$ with respect to x. The notation y' and y'' in equation 1.1c is the third way to represent derivatives; they denote the first and second derivatives of y with respect to x.

We could be more explicit in showing that a differential equation is an equation that must be solved for functions by writing equation 1.1c in the form

$$xy''(x) + y'(x) + xy(x) = 0.$$

Because this makes the differential equation more cumbersome to write, and to look at, we will refrain from using this representation, unless there is a special reason for doing so. The variable being differentiated is always dependent, depending on the other variable in the equation.

We customarily use x to denote the independent variable and y to denote the dependent variable when discussing differential equations, as in equations 1.1a,b,c,d,f. In applications, we use letters that reflect quantities that they represent. For instance, in differential equation 1.1e, x represents displacement of the mass m from its equilibrium position, and t denotes time. If temperature at points in a sphere depends only on distance from the centre of the sphere, we would use r to represent this distance, and T to represent temperature.

Differential equations can be classified in various ways, as *ordinary* or *partial*, as *linear* or *nonlinear* , and as to *order*. When a differential equation is to be solved for a function of only one variable, as in equations 1.1, it is called an **ordinary** differential equation. When y is a function of more than one variable, say x and t, and the differential equation contains partial derivatives of y with respect to x and/or t, we have a **partial** differential equation. Examples of partial differential equations are

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \tag{1.2a}$$

called the *one-dimensional diffusion equation*, and

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \tag{1.2b}$$

the *one-dimensional wave equation*. Another important partial differential equation is *Poisson's equation*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = f(x,y,z), \tag{1.2c}$$

for $V(x,y,z)$. A special case of Poisson's equation occurs when $f(x,y,z) = 0$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{1.2d}$$

It is called Laplace's equation. Chapters 1–10 deal with ordinary differential equations; Chapters 11–14 tackle partial differential equations. With this in mind, we shall refer to *ordinary differential equations* in Chapters 1–10 simply as *differential equations*.

Differential equations are also classified according to the highest order derivative contained therein.

Definition 1.1 The **order** of a differential equation is the order of the highest derivative in the equation.

The first of differential equations 1.1 is first-order, the second, third, and fifth are second-order, the fourth is fourth-order, and the last is tenth-order. Each partial differential equation 1.2a–d is second-order.

The third way to distinguish between differential equations is most important; it classifies them as *linear* or *nonlinear*. You have already encountered the concept of "linearity" in various settings. First, you will recall that a linear equation in x and y is one of the form $ax + by = c$, where a, b, and c are constants; it describes a line in the xy-plane. We say that the left side of the equation is a linear combination of x and y. The equation $Ax+By+Cz = D$, where coefficients are again constants, is linear in x, y, and z; it describes a plane in xyz-space. The left side of the equation is a linear combination of x, y, and z. We can write a vector $\mathbf{v}$ with Cartesian components (v_x, v_y, v_z) in the form $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit basis vectors in the x-, y-, and z-directions, respectively. This expresses $\mathbf{v}$ as a linear combination of the basis vectors. If $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ are vectors, the vector $\mathbf{v} = c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n$, where the c_i are constants is called a linear combination of vectors $\mathbf{v}_i$. With these ideas recalled, we now define what we mean by a linear differential equation.

Definition 1.2 An n^{th}-order differential equation in $y(x)$ is said to be **linear** if it can be expressed in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x). \quad (1.3)$$

The left side of the equation is a linear combination of $y(x)$ and its n derivatives. This is one reason for calling the equation linear; a more important reason will be discussed in Section 4.1. Coefficients $a_i(x)$ can be numbers, but they can also be functions of x (but not functions of y). Notice in particular that none of the derivatives of $y(x)$ are multiplied together, nor are they squared or cubed or taken to any other power, nor do they appear as the argument of any transcendental function. There is a function of x multiplying y, **plus** a function of x multiplying the first derivative of y, **plus** a function of x multiplying the second derivative of y, and so on, to the n^{th} derivative of y. In order that the equation be n^{th}-order, we assume that $a_n(x)$ is not identically equal to zero. It could be equal to zero at isolated values of x, but not equal to zero for all x. For example, linear first- and second-order equations are ones that can be expressed in the forms

$$a_1(x)\frac{dy}{dx} + a_0(x)y = F(x), \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x).$$

The following equations are nonlinear.

$$x\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \sin y = e^{2x}, \qquad \frac{d^3y}{dx^3}\frac{dy}{dx} - 2y = x^2.$$

The first equation is nonlinear because of the $\sin y$-term; the second is nonlinear due to the product of d^3y/dx^3 and dy/dx. Differential equations 1.1c,d,e,f are linear; equations 1.1a,b are nonlinear.

Example 1.1 Determine whether the following differential equations are linear:

$$\text{(a)}\ \sin x\frac{d^2y}{dx^2}+3e^{2x}y^2=6x \quad \text{(b)}\ y''+e^xy'+xy=\sin 2x \quad \text{(c)}\ (1+y')^{1/3}+x^2=0$$

Solution (a) The $3e^{2x}y^2$ in this equation makes the equation nonlinear, not because of e^{2x}, but because of y^2.

(b) This equation is linear; it is of form 1.3. Coefficient e^x, and $\sin 2x$, are not linear functions of x, but that is irrelevant. The left side of the equation is a linear combination of y, y' and y''.

(c) Because we can rewrite this equation in the form $\dfrac{dy}{dx}=-1-x^6$, it is linear.●

Solutions of Differential Equations

Differential equations must be solved for functions. A function is a solution of a differential equation if it satisfies the following requirement.

Definition 1.3 A function is a **solution** of a differential equation on an open interval† I if substitution of the function into the differential equation reduces the differential equation to an identity on the interval.††

What we mean by saying that the differential equation is "reduced to an identity" is that when the proposed solution is substituted into the differential equation, left and right sides of the equation become equal at every point in the interval I. For example, to show that the function $y(x)=e^{5x}-2e^{-x}+x^2+2x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=-5x^2-18x-6$$

on the interval $-\infty<x<\infty$, we substitute it into the left side of the differential equation,

$$\begin{aligned}\frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y&=(25e^{5x}-2e^{-x}+2)-4(5e^{5x}+2e^{-x}+2x+2)\\&\quad-5(e^{5x}-2e^{-x}+x^2+2x)\\&=-5x^2-18x-6.\end{aligned}$$

Since this is the right side of the differential equation, we have verified that substitution of the function into the differential equation reduces it to an identity on the interval $-\infty<x<\infty$.

† By considering differential equations on open intervals we avoid one-sided derivatives at the ends of the interval.

†† This is a text on **real** differential equations, and therefore we seek real-valued solutions. We may sometimes find it useful to introduce complex numbers, but ultimately we want real solutions.

Example 1.2 Show that $y(x) = C_1 \cos 3x + C_2 \sin 3x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} = -9y$$

on the interval $-\infty < x < \infty$ for any constants C_1 and C_2.

Solution If we substitute the function into the left side of the differential equation, we get

$$\frac{d^2y}{dx^2} = -9C_1 \cos 3x - 9C_2 \sin 3x.$$

Substitution into the right side of the equation gives

$$-9y = -9(C_1 \cos 3x + C_2 \sin 3x).$$

Since these expressions are equal for all x and any C_1 and C_2, the given function is indeed a solution of the differential equation on the interval $-\infty < x < \infty$. Because $y(x) = C_1 \cos 3x + C_2 \sin 3x$ contains two arbitrary constants, it is not just a solution, but a double-infinity of solutions. We call it a **2-parameter family of solutions.**•

Example 1.3 What is the largest interval on which $y(x) = x^{-1/2} \sin x$ is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right) y = 0?$$

Solution Substitution of the function into the left side of the differential equation gives

$$\begin{aligned} x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right) y &= x^2 \left(-\frac{\sin x}{\sqrt{x}} - \frac{\cos x}{x^{3/2}} + \frac{3 \sin x}{4x^{5/2}}\right) \\ &\quad + x\left(\frac{\cos x}{\sqrt{x}} - \frac{\sin x}{2x^{3/2}}\right) + \left(x^2 - \frac{1}{4}\right)\frac{\sin x}{\sqrt{x}} \\ &= \sin x \left(-x^{3/2} + \frac{3}{4\sqrt{x}} - \frac{1}{2\sqrt{x}} + x^{3/2} - \frac{1}{4\sqrt{x}}\right) \\ &\quad + \cos x(-\sqrt{x} + \sqrt{x}) \\ &= 0. \end{aligned}$$

Consequently the function $y(x) = x^{-1/2} \sin x$ is a solution of the differential equation and is so for all values of x for which differentiations make sense; namely, $x > 0$.•

Finding Versus Checking Solutions to Differential Equations

There is a vast difference between *checking* whether a function is a solution of a differential equation, and *finding* a solution of a differential equation. If we have a function that we believe to be a solution of a differential equation, it is relatively simple to check whether this is indeed the case. As in Examples 1.2 and 1.3, we simply substitute the function into the differential equation to see if it reduces the equation to an identity. This means that you should **never** give an incorrect solution to a differential equation. You can **always** check whether it is correct.

Finding solutions to a differential equation can sometimes be a formidable task, even an impossible one. Much of this text is devoted to finding solutions of certain types of differential equations, but the reader should realize that many equations, especially those that arise in engineering, physics, economics, etc, may not fall into these categories. Particularly troublesome are nonlinear equations; to find exact solutions, it may be necessary to resort to methods beyond the scope of this book. Alternatively, approximate solutions can be derived with the methods in Chapters 9 and 10.

Intervals of Validity for Solutions to Differential Equations

When differential equations arise in applications, intervals on which solutions are desired are usually known. For instance, differential equation 1.1b describes the shape of a hanging cable, and therefore it would be solved on the open interval $0 < x < L$, given that the ends of the cable are at $x = 0$ and $x = L$. Equation 1.1e describes the displacement of a mass on the end of a spring, and it would be solved for $t > 0$, given that the motion of the mass commences at time $t = 0$. When differential equations do not arise from applications, we often find solutions, and subsequently determine their domains of validity. For instance, it is easy to see (or check) that exponential functions $y(x) = Ce^{2x}$ are solutions of the differential equation $dy/dx = 2y$ for any value of the constant C, and they are all solutions on the interval $-\infty < x < \infty$. It sometimes happens, however, that different solutions of a differential equation are defined on different intervals. For example, elementary integration suggests that functions that satisfy the differential equation

$$\frac{dy}{dx} = \frac{1}{(x+2)^2}, \tag{1.4}$$

are of the form

$$y(x) = \frac{-1}{x+2} + C, \tag{1.5a}$$

where C is an arbitrary constant. But where is this function a solution of differential equation 1.4? Clearly the differential equation need not be considered at $x = -2$. The solution is valid on the interval $x < -2$, or the interval $x > -2$, or any subinterval of these intervals, but not on any interval that contains $x = -2$. A better description of solutions is

$$y(x) = \begin{cases} -\dfrac{1}{x+2} + C_1, & x < -2 \\ -\dfrac{1}{x+2} + C_2, & x > -2 \end{cases} \tag{1.5b}$$

where C_1 and C_2 are arbitrary (unrelated) constants. Expression 1.5b contains solutions of differential equation 1.4 on the interval $x < -2$, and solutions on the interval $x > -2$, but they are completely independent solutions. Solutions have the same form on these intervals, but they are unrelated. We often use compact notation 1.5a for solutions of differential equations like 1.4, but realize that further analysis of the differential equation and its solutions might necessitate a division of 1.5a into its component parts 1.5b.

Example 1.4 Find solutions of the differential equation

$$(x^2 - 1)\frac{dy}{dx} = x.$$

Solution When we write the differential equation in the form

$$\frac{dy}{dx} = \frac{x}{x^2 - 1},$$

it can immediately be integrated to give

$$y(x) = \frac{1}{2}\ln|x^2 - 1| + C.$$

The solution should clearly not be considered at $x = \pm 1$. It is valid on the intervals $-\infty < x < -1$, $-1 < x < 1$, and $1 < x < \infty$. As a result, the best way to write the solution is

$$y(x) = \frac{1}{2}\begin{cases} \ln(x^2 - 1) + C_1, & -\infty < x < -1 \\ \ln(1 - x^2) + C_2, & -1 < x < 1 \\ \ln(x^2 - 1) + C_3, & 1 < x < \infty. \end{cases}$$

We have an infinity of solutions of the differential equation on the interval $-\infty < x < -1$, an infinity of solutions on the interval $-1 < x < 1$, and an infinity of solutions on the interval $1 < x < \infty$.•

Implicitly Defined Solutions

In the differential equations that we have considered, x has been the independent variable and y the dependent variable. In every case, solutions have been defined explicitly; that is, solutions have been written in the form $y = y(x)$. Because functions can be defined implicitly as well as explicitly, it should not be surprising that solutions of a differential equation might be defined implicitly rather than explicitly. For example, we have plotted the curve $5y^3 + x^3 + x - 10y = 1$ in Figure 1.1. It defines three functions, one being that part of the curve above the point P, a second being the part of the curve between P and Q, and the third being the part of the curve below Q. We claim that each of these functions satisfies the nonlinear differential equation

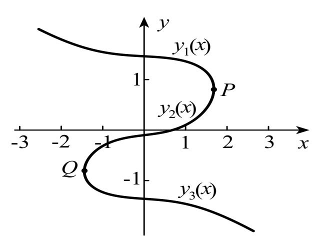


Figure 1.1

$$\frac{dy}{dx} = \frac{3x^2 + 1}{10 - 15y^2}.$$

To prove this, we differentiate the equation of the curve implicitly with respect to x,

$$15y^2\frac{dy}{dx} + 3x^2 + 1 - 10\frac{dy}{dx} = 0.$$

When this is solved for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{3x^2 + 1}{10 - 15y^2},$$

the given differential equation. All three of the functions defined by the equation $5y^3 + x^3 + x - 10y = 1$ and shown in Figure 1.1 satisfy the differential equation, but they do so on different intervals. The uppermost function $y_1(x)$ satisfies the differential equation on the interval $-\infty < x < x_P$, where x_P is the x-coordinate of P; the middle function $y_2(x)$ satisfies the differential equation on the interval $x_Q < x < x_P$, where x_Q is the x-coordinate of Q; and the lowest function $y_3(x)$ is a solution on the interval $x_Q < x < \infty$. We cannot find these functions explicitly, as it is impossible to solve $5y^3 + x^3 + x - 10y = 1$ for y in terms of x, but in Figure 1.1, we have their graphs. Each graph is called a **solution curve** of the differential equation.

The differential equation identifies the points P and Q. At these points, the slope of the curve is undefined, and this occurs when $10 - 15y^2 = 0$. Thus, y-coordinates of P and Q are $\pm\sqrt{2/3}$. Corresponding x-coordinates can be obtained by substituting these values of y into $5y^3 + x^3 + x - 10y = 1$,

$$\pm 5(2/3)^{3/2} + x^3 + x \mp 10(2/3)^{1/2} = 1.$$

These equations can be solved numerically for $x_P \approx 1.68$ and $x_Q \approx -1.44$.

We formalize these ideas in the following definition.

Definition 1.4 An equation $f(x, y) = 0$ is said to **implicitly** define a solution of a differential equation on an open interval I if there exists a function, defined implicitly by the equation, that satisfies the differential equation on I.

The above example illustrates that an equation $f(x, y) = 0$ might implicitly define more than one solution of a differential equation. Here are two further illustrations of implicitly defined solutions.

Example 1.5 Find solutions of the nonlinear differential equation

$$y\frac{dy}{dx} = -x.$$

Solution If we integrate both sides of the equation with respect to x, we obtain

$$\int y\frac{dy}{dx}\,dx = \int -x\,dx = -\frac{x^2}{2} + C.$$

Integration on the left is what might be called backwards implicit differentiation; find an expression which when differentiated with respect to x gives $y\dfrac{dy}{dx}$. Such an expression is $\dfrac{y^2}{2}$. In other words, we can write that

$$\frac{y^2}{2} = -\frac{x^2}{2} + C, \qquad \text{or,} \qquad x^2 + y^2 = 2C.$$

Since C is an arbitrary constant, but must be positive in order that the equation define functions, we replace it with D^2, in which case, solutions of the differential equation are defined implicitly by

$$x^2 + y^2 = D^2.$$

Geometrically, this equation describes a family of circles of radius D centred at the origin, one of which is shown in Figure 1.2a. Algebraically, the equation defines two continuous functions on the interval $-D \leq x \leq D$, $y(x) = \sqrt{D^2 - x^2}$ and $y(x) = -\sqrt{D^2 - x^2}$. Both are solutions of the differential equation on the interval $-D < x < D$. Notice that the endpoints $x = \pm D$ have been removed, and there are two reasons for this. Firstly, we agreed to discuss solutions of differential equations only on open intervals, and secondly, even if we had not made this agreement, these functions do not have derivatives at $x = \pm D$, and cannot therefore satisfy the differential equation at these points. The equation $x^2 + y^2 = D^2$ defines other functions on the interval $-D < x < D$, one being that in Figure 1.2b. But this function is not a solution of the differential equation on this interval because it does not have a derivative at $x = 0$.●

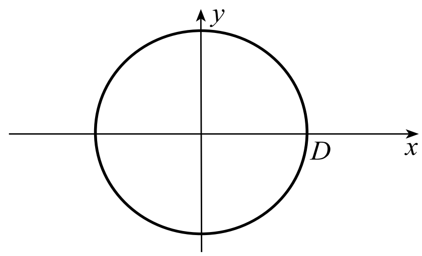


Figure 1.2a

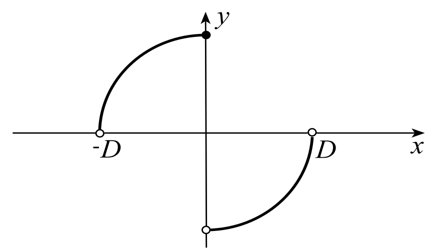


Figure 1.2b

Example 1.6 Show that the equation $y = (x + y)[\ln y + C]$, where C is a constant, implicitly defines solutions of the differential equation

$$\frac{dy}{dx} = -\frac{y^2}{x^2 + y^2 + xy}.$$

Solution Let $y(x)$ be any function that is implicitly defined by any one of the curves $y = (x+y)[\ln y+C]$. To show that its derivative satisfies the given differential equation, we use implicit differentiation. It is best (but not necessary) to isolate the constant C,

$$\frac{y}{x + y} - \ln y = C.$$

Implicit differentiation now gives

$$\frac{(x + y)dy/dx - y(1 + dy/dx)}{(x + y)^2} - \frac{1}{y}\frac{dy}{dx} = 0.$$

When we multiply by $y(x + y)^2$, we get

$$y(x + y)\frac{dy}{dx} - y^2\left(1 + \frac{dy}{dx}\right) - (x + y)^2\frac{dy}{dx} = 0.$$

We now solve this for dy/dx,

$$\frac{dy}{dx} = \frac{y^2}{y(x + y) - y^2 - (x + y)^2} = -\frac{y^2}{x^2 + xy + y^2}.$$

In Figure 1.3 we have shown plots of the curves $y = (x + y)[\ln y + C]$ for values $C = 1$ and $C = 2$. Any function defined by any one of these curves implicitly defines a solution of the differential equation on some interval. The interval could be $x < 0$, or $-\infty < x < \infty$, depending first on the choice of constant C, and then on the particular choice of curve resulting from that C.•

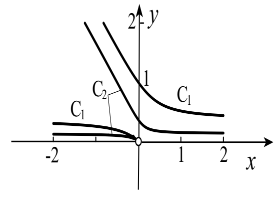
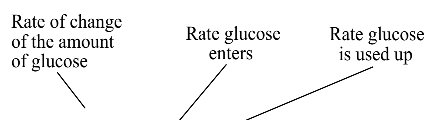


Figure 1.3

In these two examples, we used graphs to illustrate implicitly defined solutions of differential equations. To determine algebraically when an equation implicitly defines functions, we can use the "implicit function theorem" from calculus. It is discussed in Exercise 47 at the end of this section.

Boundary and Initial Conditions

When differential equations occur in applications, they are usually accompanied by subsidiary conditions called *initial* or *boundary conditions.* For example, suppose a drug such as glucose is administered intravenously into the bloodstream of a hospital patient at a constant rate R units per unit time. It is often assumed that the body uses the glucose up at a rate proportional to the amount present at that time. If $A(t)$ is the amount of glucose in the blood as a function of time t, then its derivative dA/dt is the time rate of change of the amount of glucose in the blood. It must be equal to the rate R at which glucose enters the bloodstream less the rate kA at which the body uses it up,



$$\frac{dA}{dt} = R - kA, \tag{1.6}$$

where $k > 0$ is a constant. This is a differential equation for $A(t)$. If the initial amount of glucose in the blood at time $t = 0$ is A_0, then $A(t)$ must also satisfy $A(0) = A_0$. This is an **initial condition** that solution(s) of the differential equation must also satisfy. In other words, the real problem is to find solution(s) of differential equation 1.6 that also satisfy the initial condition $A(0) = A_0$,

$$\frac{dA}{dt} = R - kA, \quad t > 0, \tag{1.7a}$$

$$A(0) = A_0. \tag{1.7b}$$

This is the form in which analysts in applied areas find differential equations — the differential equation is accompanied by subsidiary conditions that express other requirements of solutions. We call problem 1.7 an **initial-value problem**. It is not difficult to verify that solutions of the differential equation are

$$A(t) = \frac{R}{k} + Ce^{-kt}, \tag{1.8}$$

for any constant C. When we impose the initial condition, we obtain

$$A_0 = A(0) = \frac{R}{k} + C.$$

Thus, $C = A_0 - R/k$, and a solution of the initial-value problem is

$$A(t) = \frac{R}{k} + \left(A_0 - \frac{R}{k}\right) e^{-kt}. \tag{1.9}$$

This was your first exposure to mathematical modelling with differential equations. We turned the physical assumption that the body uses glucose up at a rate proportional to the amount present into differential equation 1.6. To complete the model we added the initial condition $A(0) = A_0$, resulting in initial-value problem 1.7.

Mathematical modelling is one of the most important areas of mathematics. It is used to describe oscillations of physical systems such as machinery, buildings during storms, and currents in electical networks. Models have been constructed to describe the spread of diseases, the conduction of heat, and the diffusion of chemicals. The list of applications of mathematical models is vast, and ever growing. We will discuss mathematical modelling in numerous applications of differential equations throughout the rest of this text. We may not always mention that we are doing mathematical modelling, but the reader will recognize this.

A second situation that may already be familiar to some readers is that of a mass M suspended from a spring with constant k. If the mass is also subject to air drag that is proportional to its velocity, then displacement $x(t)$ of the mass from its equilibrium position as a function of time t must satisfy the differential equation

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0, \tag{1.10a}$$

where $\beta > 0$ is a constant. (Don't worry if you have never seen this equation before. We will derive and analyze it from every possible viewpoint in Chapter 5.) Accompanying the differential equation will be two initial conditions specifying the position and velocity of the mass at some initial time t_0,

$$x(t_0) = x_0, \quad x'(t_0) = v_0. \tag{1.10b}$$

The initial-value problem then consists of finding solution(s) of differential equation 1.10a that also satisfy initial conditions 1.10b.

Equation 1.1d, which is used to determine the deflection of a beam, is normally accompanied by four conditions, two at each end $x = 0$ and $x = L$ of the beam. Examples are

$$y(0) = y(L) = 0, \quad y''(0) = y''(L) = 0.$$

They are called **boundary conditions**. Differential equation 1.1d together with these boundary conditions is called a **boundary-value problem**. It is straightforward to verify that the function

$$y(x) = C_1e^{kx} + C_2e^{-kx} + C_3 \sin kx + C_4 \cos kx - \frac{1}{k^4}, \tag{1.11}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, satisfies differential equation 1.1d. When the boundary conditions are applied, these constants must satisfy the four equations

$$\begin{aligned}
0 &= C_1 + C_2 + C_4 - 1/k^4, \\
0 &= C_1 e^{kL} + C_2 e^{-kL} + C_3 \sin kL + C_4 \cos kL - 1/k^4, \\
0 &= C_1 + C_2 - C_4, \\
0 &= C_1 e^{kL} + C_2 e^{-kL} - C_3 \sin kL - C_4 \cos kL.
\end{aligned}$$

The solution is

$$C_1 = \frac{1}{2k^4(1+e^{kL})}, \quad C_2 = \frac{e^{kL}}{2k^4(1+e^{kL})}, \quad C_3 = \frac{\csc kL - \cot kL}{2k^4}, \qquad C_4 = \frac{1}{2k^4}.$$

With these, we have a solution of the boundary-value problem.

Example 1.7 Prove that $y = (C_1 + C_2 x)e^{-2x} + 1/4$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 1$$

for all x. Find a solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = -1$.

Solution When we substitute $y = (C_1 + C_2 x)e^{-2x} + 1/4$ into the left side of the differential equation, we get

$$\begin{aligned}
\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y &= (4C_1 - 4C_2 + 4C_2 x)e^{-2x} + 4(C_2 - 2C_1 - 2C_2 x)e^{-2x} \\
&\quad + 4[(C_1 + C_2 x)e^{-2x} + 1/4] = 1.
\end{aligned}$$

Because these calculations are valid for all x, the given function is a solution of the differential equation for all x. The initial conditions require

$$1 = y(0) = C_1 + \frac{1}{4}, \qquad -1 = y'(0) = C_2 - 2C_1.$$

These imply that $C_1 = 3/4$ and $C_2 = 1/2$, and therefore a solution of the initial-value problem is $y = [(3 + 2x)e^{-2x} + 1]/4$.•

Example 1.8 Find a solution of the differential equation $(x-1)\dfrac{dy}{dx} = 1$ that also satisfies the condition $y(-1) = 5$, and one that satisfies $y(4) = 6$.

Solution If we write

$$\frac{dy}{dx} = \frac{1}{x-1}, \qquad \text{integration gives} \qquad y = \ln|x-1| + C.$$

This compact notation from elementary calculus, although correct, disguises the true nature of the differential equation and its solutions. Clearly the differential equation makes no sense at $x = 1$. It should be considered on the intervals $x < 1$ and $x > 1$, or any subinterval of these intervals. Solutions on these intervals are

$$y = \begin{cases} \ln(1-x) + C_1, & x < 1 \\ \ln(x-1) + C_2, & x > 1 \end{cases}.$$

This is a much more accurate description of solutions than the compact notation $y = \ln|x-1| + C$. To satisfy the condition $y(-1) = 5$, we use the solution $\ln(1-x) + C_1$.

It requires $5 = \ln 2 + C_1$, from which $C_1 = 5 - \ln 2$. A solution is therefore $y(x) = \ln(1 - x) + 5 - \ln 2$, valid on the interval $x < 1$. To satisfy the condition $y(4) = 6$, we use the solution $\ln(x - 1) + C_2$. It requires $6 = \ln 3 + C_2$, from which $C_2 = 6 - \ln 3$. A solution is therefore $y(x) = \ln(x - 1) + 6 - \ln 3$, valid for $x > 1$.●

Did you notice that in Examples 1.7 and 1.8, we did not ask for "the" solution of the initial-value problem; we asked for "a" solution. The reason for this is that we have no way of knowing whether there is only one solution, or many solutions, and the word "the" would imply one solution. In Section 1.3, we find out when we can expect an initial-value problem to have one, and only one, solution. Until then, we must not be so bold as to say that once we have "a" solution to an initial-value problem, we have "the" solution to the problem.

Families of Solutions and Singular Solutions

We claim, without justification at the moment, that every solution of differential equation 1.6 can be written in form 1.8, and every solution of equation 1.1d can be expressed as 1.11. We call expression 1.8 a one-parameter family of solutions of equation 1.6, and expression 1.11 a four-parameter family of solutions of equation 1.1d. In both cases the number of parameters (or arbitrary constants) is the same as the order of the differential equation. We might suspect a general result emerging here to the effect that every solution of an n^{th}-order differential equation is contained in an n-parameter family of solutions. For linear differential equations this is true (a fact that will be proved later), but unfortunately it is not generally true for nonlinear equations. As an illustration, consider the nonlinear equation

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2. \tag{1.12}$$

In Example 2.30, we apply standard techniques for solving differential equations to obtain the two-parameter family of solutions $y(x) = C_1 - \ln(C_2 + x)$. This two-parameter family of solutions does not, however, contain all solutions of the differential equation, because no choice of C_1 and C_2 will give the perfectly acceptable solution $y(x) = k$, where k is a constant. Constant functions are not particularly interesting, but they are nonetheless solutions that are not contained within the two-parameter family. Such solutions are called **singular solutions** of the two-parameter family. Notice that we say that a singular solution is a singular solution of the family of solutions, not a singular solution of the differential equation. We say this because sometimes by writing a family of solutions of a differential equation in a different form, the new family may contain the singular solution of the original family.

Example 1.9 Show that $y(x) = \dfrac{1}{1 + Ce^{-x}}$ is a solution of the nonlinear differential equation

$$\frac{dy}{dx} = y(1 - y)$$

on the interval $-\infty < x < \infty$ for any positive constant C. Can you see any singular solutions of this family of solutions?

Solution Since

$$\frac{dy}{dx} = \frac{-1}{(1+Ce^{-x})^2}(-Ce^{-x}) = \frac{Ce^{-x}}{(1+Ce^{-x})^2}$$

and

$$y(1-y) = \frac{1}{1+Ce^{-x}}\left(1 - \frac{1}{1+Ce^{-x}}\right) = \frac{Ce^{-x}}{(1+Ce^{-x})^2},$$

the function is indeed a solution of the differential equation. Because $C > 0$, the function is a solution on the interval $-\infty < x < \infty$. It is clear that the function $y(x) \equiv 0$ is also a solution of the differential equation, and it cannot be obtained from $(1+Ce^{-x})^{-1}$ for any value of C. Thus, $y(x) \equiv 0$ is a singular solution of the one-parameter family of solutions.•

General Solutions

We have illustrated that a solution of a differential equation that contains the same number of arbitrary constants as the order of the differential equation may or may not contain all solutions of the differential equation. In spite of this unfortunate circumstance, there do exist classes of differential equations for which a solution with the same number of arbitrary constants as the order of the equation does represent all possible solutions (linear equations have this property). This prompts us to make the following definition.

Definition 1.5 An n-parameter family of solutions of an n^{th}-order differential equation is said to be a **general solution** if it contains all solutions of the differential equation†.

Consequently, in order that a family of functions be a general solution of a differential equation, three conditions must be satisfied:

1. Each function in the family must be a solution of the differential equation.
2. The family must contain the requisite number of arbitrary constants (n for an n^{th}-order differential equation).
3. The family must contain all solutions of the differential equation.

Notice that we speak of "a" general solution of a differential equation, and not "the" general solution. The reason for this is that if there is one general solution of a differential equation, then there is an infinite number of general solutions, an infinite number of ways to express all solutions of the differential equation. For instance, two solutions of the linear differential equation $x^2y'' - 4xy' + 6y = 0$ are x^3 and x^2. It is straightforward to show that for any constants C_1 and C_2, the function $y(x) = C_1x^3 + C_2x^2$ is also a solution; it is a two-parameter family of solutions of the differential equation. We cannot yet, but in Chapter 4 we will be able to prove that every solution of the differential equation can be expressed in this form, and hence $y(x) = C_1x^3 + C_2x^2$ is a general solution. Functions $x^3 - x^2$ and $2x^3 + 4x^2$ are solutions of the differential equation, and $y(x) = C_3(x^3 - x^2) + C_4(2x^3 + 4x^2)$ is also a general solution. So also is $y(x) = C_5(3x^3 + 8x^2) + C_6(x^2 - 4x^3)$, but $y(x) = C_7(x^3 - 2x^2) + C_8(3x^3 - 6x^2)$ is not a general solution. Can you see the difference?

† Readers should be aware that not all authors agree on this definition of a general solution of a differential equation. Some do not require a general solution to contain all solutions of the differential equation. To them, an n-parameter family of solutions is a general solution.

Example 1.10 Show that $y(x) = e^x(C_1 \cos 2x + C_2 \sin 2x)$ is a two-parameter family of solutions of the linear differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

on the interval $-\infty < x < \infty$. Is it a general solution?

Solution If we substitute $y(x)$ into the left side of the differential equation,

$$\begin{aligned}\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = {} & e^x(-3C_1 \cos 2x - 4C_1 \sin 2x - 3C_2 \sin 2x + 4C_2 \cos 2x) \\ & - 2e^x(C_1 \cos 2x - 2C_1 \sin 2x + C_2 \sin 2x + 2C_2 \cos 2x) \\ & + 5e^x(C_1 \cos 2x + C_2 \sin 2x) = 0.\end{aligned}$$

This shows that $y(x)$ satisfies the differential equation, and does so for all x. We cannot be sure that this two-parameter family of solutions contains all solutions of the differential equation, and cannot therefore claim that it is a general solution. In Chapter 4, when we consider the whole topic of linear differential equations, we rectify this.●

Example 1.11 Find a general solution for the differential equation $\dfrac{d^2y}{dx^2} = xe^{-x}$.

Solution Integration of both sides of the differential equation, with integration by parts on the right, gives

$$\frac{dy}{dx} = -xe^{-x} - e^{-x} + C_1.$$

A second integration yields

$$y(x) = xe^{-x} + 2e^{-x} + C_1 x + C_2.$$

Elementary calculus assures us that inclusion of constants of integration leads to all solutions of the differential equation, and therefore we have a general solution.●

There is no simple procedure that always determines whether an n-parameter family of solutions of an n^{th}-order differential equation is a general solution. It may happen, as in Example 1.11, that the method of arriving at the n-parameter family of solutions guarantees that all solutions are captured. Unfortunately, this is the exception rather than the rule. In Example 1.7, we illustrated that $y = (C_1 + C_2 x)e^{-2x} + 1/4$ is a two-parameter family of solutions of the linear differential equation $y'' + 4y' + 4y = 1$. At this time, it is not possible for us to show that all solutions of this differential equation can be obtained by specifying values for C_1 and C_2, and hence we cannot claim to have a general solution.

In spite of the fact that we do not have a definitive procedure by which to determine whether an n-parameter family of solutions of an n^{th}-order differential equation is a general solution, there do exist classes of differential equations for which this is indeed the case. We shall certainly point them out as we encounter them. When this is the case, we need not worry about singular solutions; there cannot be any. As we have already mentioned, linear differential equations fall into this categary, and we verify this in Section 2.1 and Chapter 4.

Some differential equations are immediately solvable (or, as we often say, immediately integrable). For example, to solve a linear differential equation of the form

$$\frac{dy}{dx} = M(x), \tag{1.13}$$

where $M(x)$ is a given function, we integrate both sides of the equation with respect to x to obtain a one-parameter family of solutions

$$y(x) = \int M(x)\,dx + C. \tag{1.14}$$

Once again, elementary calculus assures us that by including the constant of integration, we have all solutions of the differential equation. Hence, this one-parameter family of solutions is a general solution of the differential equation.

This result is easily extended to linear, n^{th}-order differential equations of the form

$$\frac{d^n y}{dx^n} = M(x), \quad n \text{ a positive integer.} \tag{1.15}$$

We integrate successively n times to obtain a general solution

$$y(x) = \int \cdots \int M(x)\,dx \cdots dx + C_1 + C_2 x + \cdots C_n x^{n-1}. \tag{1.16}$$

Example 1.12 Find the solution of the initial-value problem

$$\frac{d^2y}{dx^2} = \cos 2x + x, \qquad y(0) = 0, \quad y'(0) = 2.$$

Solution Two integrations of the differential equation give a general solution

$$y(x) = -\frac{1}{4}\cos 2x + \frac{x^3}{6} + C_1 x + C_2.$$

The initial conditions at $x = 0$ require C_1 and C_2 to satisfy the equations

$$0 = y(0) = -\frac{1}{4} + C_2, \qquad 2 = y'(0) = C_1.$$

Thus, the solution of the initial-value problem is

$$y(x) = -\frac{1}{4}\cos 2x + \frac{x^3}{6} + 2x + \frac{1}{4}.$$

Notice that we called this function "the" solution of the initial-value problem rather than "a" solution. We are warranted in doing this because integration led to a general solution of the differential equation, meaning that the family contains all solutions of the differential equation. In addition, there is one, and only one, solution of the equations for C_1 and C_2. Thus, the solution obtained can be the only solution of the problem.•

Particular Solutions

When a solution of a differential equation contains no arbitrary constants, it is called a **particular solution** of the differential equation. It follows therefore that particular solutions can be obtained by assigning specific values to arbitrary constants in a family of solutions. For example, $y(x) = 5 - \ln(3 + x)$ is a particular solution of differential equation 1.12, as is $y(x) = -\ln x$, both being obtained from $y(x) = C_1 - \ln(C_2 + x)$ by specifying values for C_1 and C_2. On the other hand, the singular solution $y(x) = 10$ is also a solution, but it cannot be obtained from this two-parameter family of solutions. The function $y(x) \equiv 0$ is also a particular solution of equation 1.12, and it is a singular solution. It is called the **trivial** solution. We are invariably interested in nontrivial solutions of differential equations, but it may be important to note that a differential equation has the trivial solution.

Example 1.13 Find a particular solution of the differential equation

$$5\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2y = 4.$$

Solution In Chapter 4 we develop systematic techniques for finding particular solutions for differential equations such as this. But clearly such techniques are not needed here; a simple glance tells us that $y(x) = 2$ is a solution, and since it contains no arbitrary constants, it is a particular solution.•

We have five objectives in studying differential equations. One objective is to determine conditions that guarantee that differential equations have solutions. A second objective is to find solutions of differential equations, and we will discuss many techniques for finding solutions. Thirdly, if we cannot find solutions of a particular differential equation, we would like to glean whatever information is available about solutions. Fourthly, we want to know what properties solutions of differential equation can be expected to possess. Finally, we want to use differential equations to model a multitude of applications.

The fourth objective is especially important when it comes to applications. When differential equations arise in applications they often involve parameters (constants) of the application. We want to know how values of these parameters affect solutions. For instance differential equation 1.1e describes displacement of a mass m on the end of a spring (with constant k) subject to drag proportional to velocity (with constant β). Intuitively, values of the parameters m, β, and k determine the nature of the motion of the mass. Solutions of the differential equation should reflect these expectations. In other words, we want to understand every aspect of differential equations; conditions guaranteeing existence of solutions, techniques for finding solutions, information about solutions when they are unattainable, properties of solutions, and applications.

EXERCISES 1.1

In Exercises 1–8 determine open intervals in which each function of the family satisfies the differential equation. State whether the differential equation is linear or nonlinear.

1. $y(x) = 2 + Ce^{-x^2}$; $\quad \dfrac{dy}{dx} + 2xy = 4x$

2. $y(x) = \dfrac{x^3}{2} + Cx^3 e^{1/x^2}; \quad x^3\dfrac{dy}{dx} + (2 - 3x^2)y = x^3$

3. $y(x) = C_1 \sin 3x + C_2 \cos 3x; \quad \dfrac{d^2y}{dx^2} + 9y = 0$

4. $y(x) = \dfrac{C_1^2 e^{2x} + 1}{2C_1 e^x} + C_2; \quad \left(\dfrac{d^2y}{dx^2}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$

5. $y(x) = C_1 e^{2x} \cos(x/\sqrt{2}) + C_2 e^{2x} \sin(x/\sqrt{2}); \quad 2\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 9y = 0$

6. $y(x) = C_1 \cos 2x + C_2 \sin 2x + C_3 \cos x + C_4 \sin x; \quad \dfrac{d^4y}{dx^4} + 5\dfrac{d^2y}{dx^2} + 4y = 0$

7. $y(x) = (C_1 + C_2 x - x^2/4)e^{4x}; \quad 2\dfrac{d^2y}{dx^2} - 16\dfrac{dy}{dx} + 32y = -e^{4x}$

8. $y(x) = C_1 \cos(2\ln x) + C_2 \sin(2\ln x) + 1/4; \quad x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 1$

In Exercises 9–18 determine whether the differential equation is linear or nonlinear. Make no attempt to solve the equation.

9. $2x\dfrac{d^2y}{dx^2} + x^3 y = x^2 + 5$

10. $2x\dfrac{d^2y}{dx^2} + x^3 y = x^2 + 5y$

11. $2x\dfrac{d^2y}{dx^2} + x^3 y = x^2 + 5y^2$

12. $x\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 1 - \sin x$

13. $x\dfrac{d^3y}{dx^3} + 3y\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 10\sin x$

14. $\sin y\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 10\sin x$

15. $y'' - 3y' - 2y = 9\sec^2 x$

16. $yy'' + 3y' - 2y = e^x$

17. $(1 + y')^{1/3} + x^2 = 4$

18. $y'''' + y'' - y = \ln x$

In Exercises 19–22 find a particular solution of the differential equation in Exercise 3 that satisfies the conditions.

19. $y(0) = 1,\ y'(0) = 6$

20. $y(0) = 2,\ y(\pi/2) = 3$

21. $y(\pi/12) = 0,\ y'(\pi/12) = 1$

22. $y(1) = 1,\ y(2) = 2$

In Exercises 23–27 find a general solution for the differential equation.

23. $\dfrac{dy}{dx} = 6x^2 + 2x$

24. $\dfrac{dy}{dx} = \dfrac{1}{9 + x^2}$

25. $\dfrac{d^2y}{dx^2} = 2x + e^x$

26. $\dfrac{d^2y}{dx^2} = x \ln x$

27. $\dfrac{d^3y}{dx^3} = \dfrac{1}{3x^5}$

28. (a) Find a solution $y = y(x)$ of the initial-value problem

$$\frac{dy}{dx} = \frac{x}{(1 - x^2)^{1/3}}, \qquad y(0) = 2.$$

(b) What is the largest interval on which $y(x)$ is a solution of the initial-value problem?
(c) What is the largest interval on which $y(x)$ is defined?

29. (a) Verify that $y = x^4/4$ is a solution of the initial-value problem

$$\frac{dy}{dx} = 2x\sqrt{y}, \qquad y(0) = 0.$$

(b) Find another solution of the problem.

In Exercises 30–34 show that the equation implicitly defines solutions of the differential equation. State whether the differential equation is linear or nonlinear.

30. $4x^2 + Cy^2 = 1, \quad \dfrac{dy}{dx} = \dfrac{4xy}{4x^2 - 1}$ Do this by isolating C before differentiation, and by not isolating it.

31. $y^2 = x^3/(C - x), \quad \dfrac{dy}{dx} = \dfrac{y^3 + 3x^2 y}{2x^3}$

32. $y - x + \mathrm{Sin}^{-1}(x + y) = C, \quad \dfrac{dy}{dx} = \dfrac{\sqrt{1 - (x + y)^2} - 1}{\sqrt{1 - (x + y)^2} + 1}$

33. $x^2 y^2 + y \ln(2 + \sin x) = C, \quad 2xy^2 + \dfrac{y \cos x}{2 + \sin x} + [2x^2 y + \ln(2 + \sin x)]\dfrac{dy}{dx} = 0$

34. $x^3 + y^2 - \ln|y + x^2| = C, \quad \left(\dfrac{1}{y + x^2} - 2y\right)\dfrac{dy}{dx} = 3x^2 - \dfrac{2x}{y + x^2}$

35. A curve in the one-parameter family $x^3 + y^3 = Cxy$ is called a **folium of Descartes**.

(a) Show that the equation implicitly defines solutions of the differential equation

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

(b) Find an implicit definition for the solution of the differential equation that satisfies the condition $y(1) = 1$. Plot the curve defined by this equation.

(c) Determine the interval on which the function defined implicitly in part (b) is a solution of the initial-value problem.

36. An important consideration in Section 4.4 will be to determine values of the constant m so that e^{mx} is a solution of a linear differential equation such as

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0.$$

Find two such values.

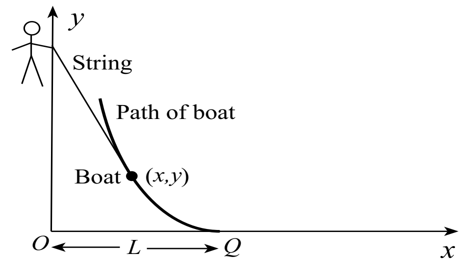
37. An important consideration in Section 4.10 will be to determine values of the constant m so that x^m is a solution of a linear differential equation such as

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$$

Find two such values.

38. (a) Jason initially at O (figure to the right) walks along the edge of a swimming pool (the y-axis) towing his sailboat by a string of length L. If the boat starts at Q and the string always remains straight, show that the equation of the curved path $y = y(x)$ followed by the boat must satisfy the initial-value problem

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}, \quad y(L) = 0.$$



(b) Find a solution of the initial-value problem.

39. (a) Show that $y(x) = 1 - (x^3 + C)^{-1}$ is a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 3x^2(y-1)^2.$$

(b) Find a singular solution.

(c) Repeat parts (a) and (b) if the differential equation is given in the form

$$\frac{1}{(y-1)^2}\frac{dy}{dx} = 3x^2.$$

40. (a) Show that $y(x) = Ce^{2x}$ is a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = 2y.$$

(b) Show that there is a particular solution that passes through any given point (x_0, y_0), and that this solution can be obtained by specifying C appropriately.

41. (a) Verify that a one-parameter family of solutions for the differential equation

$$2x\frac{dy}{dx} = y$$

is defined implicitly by the equation $y^2 = Cx$.

(b) Show that, with the exception of points on the y-axis, there is a particular solution that passes through any given point (x_0, y_0), and that this solution can be obtained by specifying C appropriately.

42. Consider the differential equation $dy/dx = 1/x^2$.

(a) Find a solution that satisfies the condition $y(1) = 1$.

(b) Find a solution that satisfies the condition $y(-1) = 2$.

(c) Is there a solution that satisfies both the conditions in parts (a) and (b)?

43. Determine whether the following functions are solutions of the differential equation $dy/dx = -x/y$ on the interval $-2 < x < 2$. If your answer is no, explain why not.

(a) $y(x) = \sqrt{4 - x^2}$

(b) $y(x) = -\sqrt{4 - x^2}$

(c) $y(x) = \begin{cases} \sqrt{4 - x^2}, & -2 < x < 0 \\ -\sqrt{4 - x^2}, & 0 \le x < 2 \end{cases}$

44. Determine whether the following functions are solutions of the differential equation $x(dy/dx) - 2y = 0$ on the interval $-\infty < x < \infty$. If your answer is no, explain why not.
(a) $y(x) = 2x^2$
(b) $y(x) = -3x^2$
(c) $y(x) = \begin{cases} -3x^2, & x < 0 \\ 2x^2, & x \geq 0 \end{cases}$

45. (a) Show that $y(x) = Cx^2$ is a one-parameter family of solutions of the differential equation in Exercise 44.
(b) If the differential equation is expressed in the form $\dfrac{dy}{dx} = \dfrac{2y}{x}$, are the functions in the one-parameter family $y(x) = Cx^2$ still solutions?

46. Results of this section would seem to suggest that differential equations always have an infinite number of solutions. Can you find a differential equation that has exactly one solution? Can you find a differential equation that has no solutions?

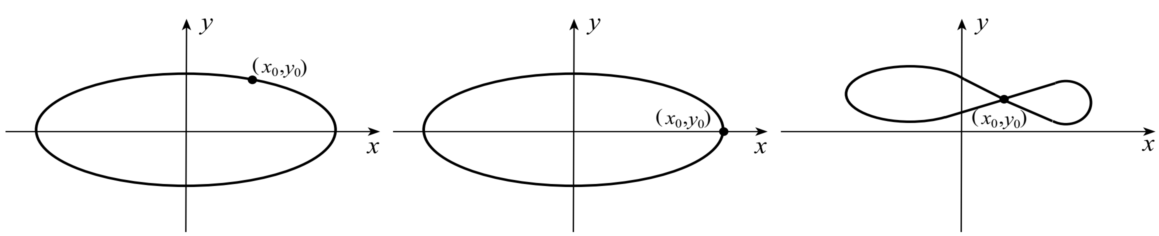
47. In this exercise, we discuss the "implicit function theorem" from calculus. It states that: An equation $F(x, y) = 0$ defines a differentiable function $y(x)$ in some open interval $|x - x_0| < \delta$ around x_0 if x_0 and $y_0 = y(x_0)$ satisfy $F(x_0, y_0) = 0$ and $\dfrac{\partial F(x_0, y_0)}{\partial y} \neq 0$. A proof of this result can be found in advanced calculus books. We illustrate its graphical interpretation here. Suppose the curve defined by the equation $F(x, y) = 0$ is as shown in the left figure below. It is clear that the curve defines a function in an interval around x_0. Its derivative can be found by implicit differentiation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

When this is solved for dy/dx at $x = x_0$, the derivative is

$$\frac{dy}{dx}_{|x_0} = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)}.$$

Notice that the requirement $F_y(x_0, y_0) \neq 0$ of the theorem ensures that the denominator of this expression does not vanish. The curves in the middle and right figures do not define y as a function of x in an open interval around x_0, let alone differentiable ones. For the middle figure, $F_y(x_0, y_0)$ would be equal to zero, and for the right figure $F_y(x_0, y_0)$ would not exist.



(a) Show that $F(x, y) = x^2 + y^2 - 4$ is an example of the left figure above provided that $-2 < x < 2$.
(b) Show that $F(x, y) = x^2 + y^2 - 4$ is an example of the middle figure when $x_0 = \pm 2$.
(c) Show that $F(x, y) = y^2 + x^2(x^2 - 1)$ is an example of the right figure when $x_0 = 0$.
(d) Show that $F(x, y) = y^3 - x$ with $x_0 = 0$ defines a function in an interval around x_0, but not a differentiable one.